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1992 J. Phys. A: Math. Gen. 25 2641

(http://iopscience.iop.org/0305-4470/25/9/033)

J. Phys. A: Math. Gen. **25** (1992) 2641–2648. Printed in the UK

On Verlinde's formula for the dimensions of vector bundles on moduli spaces

J S Dowker

Department of Theoretical Physics, Schuster Laboratory, University of Manchester, Manchester M13 9PL, UK

Received 10 October 1991

Abstract. Explicit expressions are given for the dimensions of vector bundles associated with the SU(2) Wess–Zumino–Witten model according to Verlinde's formula.

1. Introduction

Verlinde (1988) has given a formula for the dimension of a certain vector bundle, $V_{g,n}$, over the moduli space of an n-punctured Riemann surface, Σ, of genus g.

The formula has been used by Killingback (1990) and Witten (1991) to obtain the volume, $|\mathcal{R}_\Sigma|$, of the moduli space $\mathcal{R}_\Sigma = \mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ of flat G-bundles over Σ or, equivalently, of the moduli space of semi-stable holomorphic bundles over Σ.

In connection with the SU(2) Wess–Zumino–Witten model, Thaddeus (1990) has employed the formula to discuss the complete cohomology of such a moduli space. Explicit algebraic results were obtained in terms of sums of powers of cosecs. These have been encountered previously (Dowker and Banach 1978, Dowker 1989) and we wish here to make some further, elementary remarks and to draw attention to some existing results in the literature.

2. Verlinde's formula

The SU(2) Wess–Zumino–Witten model, at level k, is a sort of truncated and discretized SU(2) theory containing only those spins, j that satisfy $0 \leqslant j \leqslant k/2$. The compact, genus-$g$ Riemann surface, Σ is marked at n points, x_i, with irreducible representations of SU(2) labelled by their dimensions l_i. The vector bundle $V_{g,n}$ is denoted by $V(g; x_i, l_i)$ and Verlinde's formula can be written

$$\dim V(g; x_i, l_i) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{\prod_{i=1}^{n} \chi_{l_i}(\theta_m)}{(\sin\theta_m)^{2g-2}} = d(g; l_i) \qquad (1)$$

where $\chi_l(\theta) = \sin l\theta / \sin\theta$ is the SU(2) character of the l-representation and $\theta_m = m\pi/(k+2)$. Setting any l_i to unity removes that marked point. We also note the relation

$$\prod_{i=1}^{n} \chi_{l_i}(\theta_m) = \prod_{i=1}^{n} \frac{\sin\theta_i}{\sin\theta_m} \prod_{i=1}^{n} \chi_m(\theta_i) \qquad (2)$$

0305-4470/92/092641+08$04.50

where $\theta_i = l_i\pi/(k+2)$.

Witten (1991) immediately takes the 'classical' limit, $k \to \infty$, of (1) but it is possible to give an explicit expression for any k. We shall be contented with giving the two-pointed form $d(g; l_1, l_2)$.

Setting $m \to k+2-m$ in (1) shows that there is no mixing of fermions and bosons. That is, either both of l_1 and l_2 are even, or both are odd. Then, rewriting the product of sines, we have

$$d(g; l_1, l_2) = \frac{1}{2}\left(\frac{k+2}{2}\right)^{g-1}\left(S_g\left(k+2, \frac{(l_1-l_2)}{2}\right) - S_g\left(k+2, \frac{(l_1+l_2)}{2}\right)\right) \quad (3)$$

in terms of the twisted cosec sums

$$S_g(p,r) = \sum_{l=1}^{p-1} \cos\left(\frac{2\pi r l}{p}\right) \operatorname{cosec}^{2g}\left(\frac{\pi l}{p}\right). \quad (4)$$

Because of the mod p periodicity in r it is convenient to restrict r to the range $0 \leqslant r \leqslant p-1$ and, if necessary, to adjust $(l_1-l_2)/2$ and $(l_1+l_2)/2$. We also note the symmetry $S_g(p, p-r) = S_g(p,r)$ and set $l_1 \geqslant l_2$ without loss of generality. Symmetrization on the l_i can always be performed at the end, if desired.

Various incidental identities can be found from simple trigonometric relations. For example, the expansion

$$\sin^{2q} x = \sum_{n=0}^{q} A_q^n \cos 2nx$$

with

$$A_q^n = \frac{(-1)^n}{2^{2q-1}}\binom{2q}{q-n} \qquad \text{for } n > 0$$

and

$$A_q^0 = \frac{1}{2^{2q}}\binom{2q}{q}$$

leads to

$$S_{g-q}(p,0) = \sum_{r=0}^{q} A_g^r S_g(p,r)$$

simple cases of which are

$$S_{g-1}(p,0) = \tfrac{1}{2}\left(S_g(p,0) - S_g(p,1)\right)$$

and

$$S_{g-2}(p,0) = \tfrac{1}{8}\left(3S_g(p,0) - 4S_g(p,1) + S_g(p,2)\right).$$

It is interesting to remark that the sums $S_g(p,0)$ are the values of the Laplacian ζ-function on the discretized circle, $(2\pi/p)\mathbf{Z}_p$, at the negative integers. The relevant eigenvalues being $\sin^2(\pi l/p)$, with appropriate normalization.

Incidentally, for twisted boundary conditions on the discrete circle, the eigenvalues are $\sin^2(\pi(l+\alpha)/p)$, $(0 < \alpha \leqslant 1)$. It would be nice to relate the twisted sums, $S_g(p,r)$, to the twisted ζ-function by some discrete ζ inversion formula.

3. Evaluation of the sums

Some history of sums like (4) has been given in Dowker (1989) and further comments can be found in the appendix.

A simple contour argument allows S_g to be rewritten, for $g \geqslant 1$, as

$$S_g(p,r) = \frac{\mathrm{i}p}{2\pi} \oint_\Gamma \frac{\mathrm{d}z}{\sin^{2g} z} \frac{\cos(2p\delta z)}{\sin pz} \tag{6}$$

where $\delta = r/p - \frac{1}{2}$ and Γ is a small clockwise loop around the origin.

Being a residue, the polynomial character of S_g is immediately apparent. The evaluation is carried out in Dowker (1987). More recently, Zagier (1990, unpublished letter) has also discussed the untwisted ($r = 0$) case and Thaddeus (1990) has similarly evaluated the sum relevant for a twisted SO(3) bundle ($r = p/2$ for p even). The method is similar to our own.

Although S_g can be written as a generalized (higher-order) Bernoulli polynomial (defined in Nörlund 1924, for example), it is often more convenient to exhibit the structure in terms of polynomials in p and r. An expansion of the integrand in (6) yields

$$S_g(p,r) = (-1)^{g+1} \sum_{k=0}^{g} \frac{1}{(2k)!(2g-2k)!} D^{(2g)}_{2g-2k} D_{2k}(\delta) p^{2k} \tag{7}$$

where the coefficients $D^{(2g)}_{2g-2k}$ are given in terms of the higher-order Bernoulli polynomials $B^{(m)}_n(x)$ (Nörlund 1924) by

$$D^{(2g)}_{2g-2k} = 2^{2g-2k} B^{(2g)}_{2g-2k}(g)$$

and the $D_{2k}(\delta)$ are even polynomials of degree $2k$ in δ,

$$D_{2k}(\delta) = 2^{2k} B_{2k}(\delta')$$

where $\delta' = \delta + \frac{1}{2} = r/p$ and B_n is a standard Bernoulli polynomial. The quantity $D_{2k}(\delta)p^{2k}$ is thus a homogeneous bipolynomial of degree $2k$ in p and r. How the resulting total expression (7) is organized depends on the use to be made of it.

Interest attaches to the infinite p limit. Using the elementary cases $B^{(2g)}_0 = 1$ and $B_{2g}(0) = B_{2g}$ it is easily shown that

$$S_g(p,r) \to (-1)^{g+1} \frac{(2p)^{2g}}{(2g)!} B_{2g} \tag{8}$$

as $p \to \infty$, assuming that r is held finite.

On the other hand we may wish to let r tend to infinity such that $r/p \to \alpha$. Then,

$$S_g(p,\alpha p) \to (-1)^{g+1} \frac{(2p)^{2g}}{(2g)!} B_{2g}(\alpha). \tag{9}$$

We have used this before (Dowker and Jadhav 1989a,b) where α had the interpretation of an Aharonov–Bohm flux.

Some explicit, i.e. numerical, untwisted forms are given in Dowker and Banach (1978), Dowker (1989) and in Stanley (1979), which reference was unknown to me until recently. The twisted expressions can be obtained from the formulae in Dowker (1987a,b).

4. Generating functions

A generating function for the untwisted sums is effectively given in Fisher (1969) (see our appendix) and one can be found in Stanley (1979). This latter has also been derived by Zagier (1990, unpublished letter). We will develop a formula directly from the contour integral (6). Simple algebra yields

$$
\begin{aligned}
S(\zeta;p,r) &= \sum_{g=1}^{\infty}\sin^{2g}(\zeta/p)S_g(p,r) \\
&= \frac{\mathrm{i}}{2\pi}\oint_\Gamma \mathrm{d}z\,\frac{\cos(2z\delta)}{\sin z}\,\frac{\sin^2(\zeta/p)}{\sin((z-\zeta)/p)\sin((z+\zeta)/p)}
\end{aligned}
\tag{10}
$$

under the condition $|\sin(\zeta/p)| < |\sin(z/p)|$ which implies that the poles at $z = \pm\zeta$ are inside the contour. Hence, shrinking Γ, one finds the generating function

$$
S(\zeta;p,r) = 1 - p\frac{\cos(2\zeta\delta)}{\sin\zeta}\tan(\zeta/p) \tag{11}
$$

with $\delta = r/p - \frac{1}{2}$.

5. Use of the sums

In this section we use the sums to discuss the questions outlined in section 2. If $p, = k+2$, is allowed to tend to infinity (the classical limit) we will recover the results of Killingback and Witten. Restricting attention for the time being to the zero pointed case, these authors show that the volume of the moduli space is given by

$$
|\mathcal{R}_\Sigma| = \lim_{k\to\infty} k^{-3g+3}d(g). \tag{12}
$$

From (1) we have

$$
d(g) = \left(\frac{k+2}{2}\right)^{g-1} S_{g-1}(p;0) \tag{13}
$$

and, using the limit of $S(p;r)$ found in section 4, it is readily shown that

$$
|\mathcal{R}_\Sigma| = \frac{2^{g-1}}{(2g-2)!}|B_{2g-2}| \tag{14}
$$

agreeing with the authors just cited.

This can be expressed in generating function terms. For the scaled dimensions,

$$
\bar{d}(g;l_i) \equiv \left(\frac{2}{k+2}\right) d(g;l_i)
$$

a generating function is defined, in the zero pointed case, by

$$
\begin{aligned}
\bar{d}(\zeta) &= \sum_{g=2}^{\infty}\sin^{2g-2}(\zeta/(k+2))\bar{d}(g) \\
&= \sum_{g=1}^{\infty}\sin^{2g}(\zeta/(k+2))S_g(k+2;0) = 1-(k+2)\tan(\zeta/(k+2))\cot\zeta.
\end{aligned}
\tag{15}
$$

Then simple algebra produces,

$$\sum_{g=2}^{\infty} 2^{g-1}\zeta^{2g-2}|\mathcal{R}_\Sigma| = 1 - \frac{\zeta}{\tan\zeta} \tag{16}$$

in agreement with the previous result, (14).

The two-pointed expression (3) is explicitly written

$$d(g;l_1,l_2) = (-1)^{g+1}\frac{1}{2}\left(\frac{k+2}{2}\right)^{g-1}\sum_{k=0}^{g}\frac{1}{(2k)!(2g-2k)!} \times D^{(2g)}_{2g-2k}\left(B_{2k}(\delta^-) - B_{2k}(\delta^+)\right)(2p)^{2k} \tag{17}$$

where $\delta^- = (l_1 - l_2)/2p$ and $\delta^+ = (l_1 + l_2)/2p$.

In the classical limit we have to decide what happens to l_1 and l_2. If we say that $l_i\pi/(k+2) \to \theta_i$ then the leading behaviour is

$$d(g;l_1,l_2) \to k^{3g-1}\frac{2^g}{(2g)!}\left(B_{2g}\left((\theta_1-\theta_2)/2\pi\right) - B_{2g}\left((\theta_1+\theta_2)/2\pi\right)\right) \tag{18}$$

and the volume of moduli space, $|\mathcal{R}_\Sigma(\theta_1,\theta_2)|$, is the coefficient of k^{3g-1} in this expression.

The one-pointed expressions can be found by setting $l_2 = 1$ in (17) and $\theta_2 = \pi/p \to 0$ in (18). Thus, expanding the B_{2g} in (18) and using $B'_{2g}(x) = 2gB_{2g-1}(x)$, one finds the leading term

$$d(g;l_1) \to k^{3g-2}\frac{(-1)^g}{(2g-1)!}B_{2g-1}(\theta_1/2\pi)$$

where l_1 is odd. (The other terms in (17) contribute lower powers of k.)

The volume $|\mathcal{R}_\Sigma(l_1)|$ is the same as that derived by Witten who employs the Hurwitz ζ-function, the values of which at the negative integers are just the Bernoulli polynomials, (as shown, essentially, by Hurwitz (1882) himself.)

Acknowledgment

I would like to thank Michael Thaddeus for providing a copy of Zagier's letter.

Appendix

In this appendix we give some further history of sums like (4) and also some ancillary material that may be useful.

Eisenstein (1847) defines the sums

$$(n,x) = \sum_{m=-\infty}^{\infty}\frac{1}{(x+m)^n} \tag{19}$$

and gives several results and relations for them.

Euler had shown that $(2,x) = \pi^2 \operatorname{cosec}^2(\pi x)$ and $(4,x) = \pi^4(\operatorname{cosec}^4(\pi x) - \frac{2}{3}\operatorname{cosec}^2(\pi x))$ by purely trigonometrical means.

In terms of the Epstein ζ-function

$$Z(s,x) = \sum_{m=-\infty}^{\infty} \frac{1}{|x+m|^s} \tag{20}$$

$$Z(2n,x) = (2n,x) \qquad \text{and} \qquad \frac{\partial Z(2n,x)}{\partial x} = -2n(2n+1,x). \tag{21}$$

The relation of Z to the Hurwitz ζ-function is

$$Z(s,x) = \zeta_R(s,x) + \zeta_R(s,1-x) \tag{22}$$

$$\frac{\partial Z(s,x)}{\partial x} = -s\big(\zeta_R(1+s,x) - \zeta_R(1+s,1-x)\big). \tag{23}$$

Another way of writing (10) and (11) is

$$\sin^2\left(\frac{\zeta}{p}\right) \sum_{m=1}^{p-1} \frac{\cos(2\pi rm/p)}{\sin^2(\pi m/p) - \sin^2(\zeta/p)} = 1 - p\frac{\cos(2\zeta\delta)}{\sin\zeta}\tan\left(\frac{\zeta}{p}\right) \tag{24}$$

which is a standard summation (e.g. Wahba (1968), Hansen (1975)) and can be proved directly using partial fractions (see later) as well as in many other ways.

A further generalization of (24) is equivalent to the generating function given by Fisher (1969). The following derivation is given for variety. A known formula is (Bromwich 1926):

$$4^p \prod_{m=0}^{p-1} \left(\sin^2\left(\pi(m+\alpha)/p\right) + \sinh^2\gamma\right) = 2\big(\cosh(2p\gamma) - \cos(2\pi\alpha)\big). \tag{25}$$

Taking logs, and differentiating with respect to γ, gives

$$\sum_{m=0}^{p-1} \frac{1}{\sin^2\left(\pi(m+\alpha)/p\right) + \sinh^2\gamma} = 2p\frac{\sinh(2p\gamma)}{\sinh(2\gamma)}\frac{1}{\cosh(2p\gamma) - \cos(2\pi\alpha)} \tag{26}$$

which generalizes (24) for $r = 0$ (if $\gamma = \mathrm{i}\zeta/p$) and is another standard formula.

The generating function given by Fisher (1969) follows on setting $\gamma = 0$ and differentiating

$$\sum_{m=0}^{p-1} \ln\sin\left(\pi(m+\alpha)/p\right) = (1-p)\ln 2 + \ln\sin\pi\alpha \tag{27}$$

with respect to α to produce sums of the form

$$\sum_{m=0}^{p-1} \operatorname{cosec}^{2n}\big(\pi(m+\alpha)/p\big).$$

This means, for example, that we can evaluate the ζ-function on the discrete circle, mentioned earlier, at positive, as well as at negative, integers.

A general physical interpretation of the formulae (24) and (26) is in terms of image sums and they commonly occur in many areas (e.g. Lukosz 1973, Smith 1990).

Relatedly, the summation (24) is used in discussing the expansion of the heat kernel on a factor space of the upper-half-plane U, in the presence of elliptic fixed points. Donnelly (1979) showed that the integrated heat kernel on $\mathcal{M} = U/\Gamma$ is given by

$$K(t) = \frac{1}{(4\pi t)^{3/2}} \mathrm{e}^{-t/4} \left[4|\mathcal{M}| \int_0^\infty \mathrm{d}x \frac{x\mathrm{e}^{-x^2/t}}{\sinh x} + 4\pi t \sum_{\gamma^p = \mathrm{id}} \frac{1}{p} \sum_{m=1}^{p-1} \int_0^\infty \mathrm{d}x \, \mathrm{e}^{-x^2/t} \frac{\cosh x}{\sin^2(\pi m/p) + \sinh^2 x} \right]. \qquad (28)$$

The first term is the integral over $\mathcal{M}$ of the heat kernel on U and the second is the effect of the elliptic fixed points, $\gamma^p = \mathrm{id}$.

The asymptotic expansion of $K(t)$ in powers of t then follows easily after performing the sum over m. (Actually, Donnelly does not carry out this procedure. The calculation can be found in an appendix in Balazs and Voros, 1986.) It is possible to generalize the analysis to the twisted case $r \neq 0$ (Dowker, unpublished).

An alternative form to (24) (or (26)) is provided by

$$\frac{1}{p} \sum_{m=0}^{p-1} \frac{1}{(1 - 2t\cos(2\pi m/p) + t^2)} = \frac{1 + t^p}{(1 - t^2)(1 - t^p)} \qquad (29)$$

which can be verified algebraically by factoring the denominator on the right into linear factors involving the pth roots of unity, expanding the fraction into a sum of partial fractions and then recombining these in pairs.

The left-hand side of (29) is the Molien generating function, $M(t, Z_p)$, for the cyclic group Z_p considered as a subgroup of SO(2).

Generally, for any finite-dimensional $N \times N$-matrix group, Γ,

$$M(t, \Gamma) = \frac{1}{|\Gamma|} \sum_{A \in \Gamma} \frac{1}{\det(1 - \mathrm{t}A)} = \sum_{l=0}^{\infty} d_l t^l \qquad (30)$$

where d_l is the number of linearly independent, invariant homogeneous polynomials of degree l in the N variables acted upon the elements of Γ (e.g. Burnside 1911).

Stanley (1979) derives (29) by constructing the ring of invariant polynomials from first principles.

A standard example of (30) is for Γ a finite-dimensional subgroup of SO(3). Then if we ask for the number of independent, invariant *spherical harmonics* of degree l one finds that it is given in terms of the generating function $H(t, \Gamma) = (1 - t^2) M(t, \Gamma)$, for $\Gamma \subset \mathrm{SO}(3)$. (Polya and Meyer 1949, Meyer 1954).

We remark that in all these formulae we see the SU(2) character generating function (essentially a dual transformation on the representation labels),

$$F(\theta, \gamma) = \sum_{l=1}^{\infty} \chi_l(\theta) \mathrm{e}^{2\gamma l} = \frac{1}{2} \frac{1}{\cosh(2\gamma) - \cos\theta}.$$

Polyhedral harmonics have been early considered by Poole (1932), Laporte (1948) and others. For $G \approx$ SO(4) (or spin(4)) we can use $G \approx$ SU(2)×SU(2) to give a theory of 'polyhedral hyperspherical harmonics' and an extension to spinor-valued objects is possible.

Finally we just mention that similar sums occur in the theory of the Hirzebruch signature (Hirzebruch 1968, Zagier and Hirzebruch 1974, Donnelly 1977) especially the notion of the 'defect' and its relation to number theory.

References

Balazs N L and Voros A 1986 *Phys. Rep.* **143** 109
Bromwich T J I'A 1926 *Theory of Infinite Series* (London: Macmillan)
Burnside W 1911 *Theory of Groups of a Finite Order* (Cambridge: Cambridge University Press)
Donnelly H 1979 *Illinois J. Math.* **23** 485
—— 1977 *Topology* **16** 1
Dowker J S 1987a *Phys. Rev.* D **36** 3095
—— 1987b *Phys. Rev.* D **36** 3742
—— 1989 *J. Math. Phys.* **30** 770
Dowker J S and Banach R 1978 *J. Phys. A: Math. Gen.* **11** 2255
Dowker J S and Jadhav S P 1989 *Phys. Rev.* D **39** 1196, 2368
Eisenstein F 1847 *J. Math. (Crelle)* **35** 153
Fisher M 1969 *SIAM Rev.* **13** 116 (solution of problem 69-14)
Hansen E 1975 *A Table of Series* (Englewood Cliffs, NJ: Prentice Hall)
Hirzebruch F 1968 *Ann. Math. Stud.* **70** 3
Hurwitz A 1882 *Z. Math. Phys.* **27** 86
Killingback T P 1991 *Phys. Lett.* **260B** 303
Laporte O 1948 *Z. Naturf.* **3a** 447
Lukosz W 1973 *Z. Phys.* **262** 327
Meyer B 1954 *Can. J. Math.* **6** 135
Nörlund N E 1924 *Differenzrechnung* (Berlin: Springer)
Polya G and Meyer B 1948 *C. R. Acad. Sci.* **228** 28
Poole E G C 1932 *Proc. London. Math. Soc.* **33** 435
Smith A G 1990 *Formation and Evolution of Cosmic Strings* ed S W Hawking, G W Gibbons and T Vachaspati (Cambridge: Cambridge University Press)
Stanley R P 1979 *Bull. Am. Math. Soc.* **1** 475
Thaddeus M 1990 Conformal field theory and the cohomology of the moduli space of stable bundles *Preprint* Mathematics Institute, Oxford
Verlinde E 1988 *Nucl. Phys.* B **300** 360
Wahba G 1968 *SIAM Rev.* **11** 409 (Problem 68-4)
Witten E 1991 On quantum gauge theories in two dimensions *Preprint* Princeton IASSNS-HEP-91/3
Zagier D 1990 unpublished letter
Zagier D and Hirzebruch F 1974 *The Atiyah–Singer Index Theorem and Elementary Number Theory* (Boston: Publish or Perish)